1. Use Simpson's rule with $n = 4$ to estimate

$$\tan^{-1}(2) = \int_0^2 \frac{1}{1+x^2} dx.$$

Solution: Since we are using 4 steps, $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. Our function is $f(x) = \frac{1}{1+x^2}$. Thus we have

$$\begin{aligned} &\frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + f(2)\right] \\ &= \frac{1}{6}\left[1 + 4 \cdot \frac{4}{5} + 2 \cdot \frac{1}{2} + 4\frac{4}{13} + \frac{1}{5}\right] \\ &= \frac{1}{6}\left[1 + \frac{16}{5} + 1 + \frac{16}{13} + \frac{1}{5}\right]. \end{aligned}$$

2. Evaluate the improper integral

$$\int_3^5 \frac{1}{\sqrt{x-3}} dx.$$

Solution: Recall that the given integral is improper because

$$\lim_{t \to 3^+} \frac{1}{\sqrt{x-3}} = \infty.$$

Therefore,

$$\int_3^5 \frac{1}{\sqrt{x-3}} dx = \lim_{t \to 3^+} \int_t^5 \frac{1}{\sqrt{x-3}} dx.$$

$$\begin{aligned} \int_t^5 \frac{1}{\sqrt{x-3}} dx &= \int_t^5 (x-3)^{-\frac{1}{2}}\, dx \\ &= \left.\frac{(x-3)^{\frac{1}{2}}}{\frac{1}{2}}\right|_t^5 \\ &= \left.2\sqrt{x-3}\right|_t^5 \\ &= 2\sqrt{2} - 2\sqrt{t-3} \end{aligned}$$

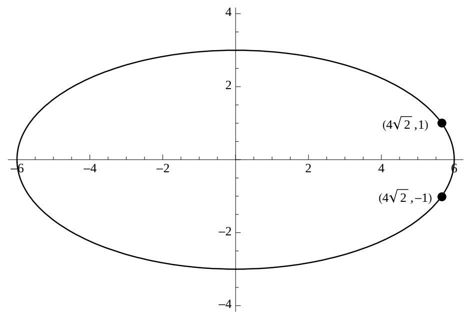
Now, $\lim_{t \to 3^+} \int_t^5 \frac{1}{\sqrt{x-3}} dx = \lim_{t \to 3^+} (2\sqrt{2} - 2\sqrt{t-3}) = 2\sqrt{2}$.

3. Which of the following integrals corresponds to the length of the shorter arc of the ellipse

$$\frac{x^2}{4} + y^2 = 9$$

(shown in the picture at right) from the point $(4\sqrt{2}, -1)$ to the point $(4\sqrt{2}, 1)$.

Solution:



This part of the curve can be written in the form $x = g(y)$. We have

$$\frac{x^2}{4} + y^2 = 9 \to \frac{x^2}{4} = 9 - y^2 \to x^2 = 4(9 - y^2) \to x = \pm 2\sqrt{9 - y^2}.$$

Since the x-values are all positive on this part of the graph, we have that this part of the graph is on the curve $x = 2\sqrt{9 - y^2}$. We get

$$\frac{dx}{dy} = \frac{-2y}{\sqrt{9 - y^2}} \to (\frac{dx}{dy})^2 = \frac{4y^2}{9 - y^2}$$

We have that the arc length of curve for $-1 \leq y \leq 1$

$$= \int_{-1}^{1} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_{-1}^{1} \sqrt{1 + \frac{4y^2}{9 - y^2}} dy = \int_{-1}^{1} \sqrt{\frac{9 + 3y^2}{9 - y^2}} dy$$

Alternatively

Noticing the options we are given we realize that we use

$$\int_{a}^{b} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

for the arc length. We see from the graph that $a = -1$ and $b = 1$. To calculate $\frac{dx}{dy}$ we use implicit differentiation.

$$\begin{aligned}
&\frac{d}{dx}(\frac{x^2}{4} + y^2) = \frac{d}{dx}(9) \\
&\Rightarrow \frac{x}{2} + 2y\frac{dy}{dx} = 0 \\
&\Rightarrow 2y\frac{dy}{dx} = \frac{x}{2} \\
&\Rightarrow \frac{dy}{dx} = \frac{x}{4y} \\
&\Rightarrow \frac{dx}{dy} = \frac{4y}{x}.
\end{aligned}$$

Note from the given equation that $\frac{x^2+4y^2}{4} = 9 \Rightarrow x^2 = 36 - 4y^2$. Therefore the arc length is given by

$$\begin{aligned}
\int_{-1}^{1} \sqrt{1 + \frac{16y^2}{x^2}}\, dy &= \int_{-1}^{1} \sqrt{1 + \frac{16y^2}{36 - 4y^2}}\, dy \\
&= \int_{-1}^{1} \sqrt{1 + \frac{16y^2}{4(9 - y^2)}}\, dy \\
&= \int_{-1}^{1} \sqrt{1 + \frac{4y^2}{9 - y^2}}\, dy \\
&= \int_{-1}^{1} \sqrt{\frac{9 - y^2 + 4y^2}{9 - y^2}}\, dy \\
&= \int_{-1}^{1} \sqrt{\frac{9 + 3y^2}{9 - y^2}}\, dy
\end{aligned}$$

4. Evaluate the improper integral

$$\int_1^\infty \frac{x}{e^{x^2/2}} dx.$$

Solution: Recall that

$$\int_1^\infty \frac{x}{e^{x^2/2}} dx = \lim_{t\to\infty} \int_1^t \frac{x}{e^{x^2/2}} dx.$$

To calculate $\int \frac{x}{e^{x^2/2}}dx$ we use u-substitution. Let $u = \frac{x^2}{2}$ then $du = x\,dx$. This implies that

$$\begin{aligned}\int \frac{x}{e^{x^2/2}}dx &= \int \frac{1}{e^u}\,du \\ &= \int e^{-u}\,du \\ &= -e^{-u} \\ &= -\frac{1}{e^{x^2/2}}\end{aligned}$$

Now, $\int_1^t \frac{x}{e^{x^2/2}}dx = -\frac{1}{e^{x^2/2}}\Big|_1^t = -\frac{1}{e^{t^2/2}} + \frac{1}{e^{1/2}}$. Note that $\lim_{t\to\infty} \frac{1}{e^{t^2/2}} = 0$. This gives us

$$\int_1^\infty \frac{x}{e^{x^2/2}}dx = \frac{1}{\sqrt{e}}.$$

5. Use Euler's method with step size 0.2 to estimate $y(0.4)$ where $y(x)$ is the solution to the initial value problem

$$y' = 10(x+y)^2, \quad y(0) = 0.$$

Solution: Here $F(x,y) = 10(x+y)^2$, $h = 0.2$ and the initial point is $(0,0)$. Therefore,

$$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2 \cdot F(0,0) = 0.$$

Now,

$$y_2 = y_1 + hF(x_1, y_1) = 0 + 0.2 \cdot F(0.2, 0) = 0.2 \cdot 10(0.2)^2 = 0.2 \cdot 0.4 = 0.08.$$

6. Find the solution of the differential equation:

$$\frac{dy}{dx} = \frac{x+1}{e^y},$$

with initial condition $y(0) = 2$.

Solution:
First we separate the variables:

$$e^y dy = (x+1)dx.$$

Integrating both sides yields

$$e^y = x^2/2 + x + C.$$

To solve for y we take the log of both sides:

$$y = \ln(x^2/2 + x + C).$$

Finally, we use the initial condition $y(0) = 2$ to solve for C.

$$2 = \ln(C) \implies C = e^2.$$

Thus

$$y = ln(x^2/2 + x + e^2).$$

Note that $ln(x^2/2 + x + e^2) = ln|x^2/2 + x + e^2|$ since $x^2/2 + x + e^2 > 0$.

7. Find the general solution of the differential equation:

$$y' - \left(\frac{1}{x}\right)y = 1 + x^2.$$

Solution: This is a first order linear differential equation which is already in standard form. We find the integrating factor:

$$I(x) = e^{\int(-1/x)dx} = e^{-\ln|x|} = (|x|)^{-1} = \frac{1}{|x|}.$$

Multiplying the equation above by $I(x) = \frac{1}{|x|}$ is the same as multiplying by $\frac{1}{x}$. we get $y'/x - y/x^2 = \frac{1+x^2}{x}$ giving us that

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x} + x.$$

This gives that

$$\frac{y}{x} = \int (1 + x^2)dx = \ln|x| + \frac{x^2}{2} + C.$$

Multiplying across by x, we get

$$y = x(\ln|x| + \frac{x^2}{2} + C).$$

8. Determine if the sequence given by $a_n = \frac{\tan^{-1}(n)}{n}$ converges or diverges, and if it converges find

$$\lim_{n\to\infty} \frac{\tan^{-1}(n)}{n}.$$

Solution:
As $n \to \infty$, $\tan^{-1}(n) \to \pi/2$. Hence the numerator of the sequence approaches $\pi/2$ while the denominator approaches $+\infty$. This means that $a_n \to 0$ as $n \to \infty$.

9. Consider the following sequences:

$$(I)\ \left\{(-1)^n \frac{n^2-1}{2^n}\right\}_{n=1}^{\infty} \qquad (II)\ \left\{(-1)^n \frac{n^2-1}{2n^2}\right\}_{n=1}^{\infty} \qquad (III)\ \left\{(-1)^n n\ln(n)\right\}_{n=1}^{\infty}.$$

Which converge, and which diverge?

Solution:
(I): By applying L'Hospital's Rule to the function $f(x) = \frac{x^2-1}{2^x}$ we can see that $\lim_{x\to\infty} f(x) = 0$. Thus $\lim_{n\to\infty} \frac{n^2-1}{2^n} = 0$. But for $n \geq 1$,

$$\frac{n^2-1}{2^n} = \left|(-1)^n \frac{n^2-1}{2^n}\right|,$$

so the sequence (I) also converges to 0.

(II): $\lim_{n\to\infty} \frac{n^2-1}{2n^2} = 1/2$, so as n grows large, the expression $(-1)^n \frac{n^2-1}{2n^2}$ oscillates

between values close to $+1/2$ (when n is even) and values close to $-1/2$ (when n is odd). Thus the sequence (II) diverges.

(III): As $n \to \infty$, $n\ln(n)$ grows arbitrarily large. The factor of $(-1)^n$ in sequence (III) makes the values oscillate between positive values of large magnitude and negative values of large magnitude. Thus the sequence (III) diverges.

10. Find the sum of the following series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n+1}}{3^n}$$

Solution: This is a geometric series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots = \begin{cases} \text{converges to} & \frac{a}{1-r} \quad \text{if } |r| < 1 \\ \text{diverges} & \quad\quad \text{if } |r| \geq 1. \end{cases}$$

(technically we should check if a_{n+1}/a_n is a constant r in order to check this.) We can identify a by calculating the first term with a_1. When $n = 1$, we get

$$a = a_1 = \frac{(-1)^1 2^{1+1}}{3^1} = -\frac{2^2}{3}.$$

When $n = 2$, we get

$$ar = a_2 = \frac{(-1)^2 2^{2+1}}{3^2} = \frac{2^3}{3^2}.$$

Now we have

$$r = \frac{a_2}{a_1} = \left(\frac{2^3}{3^2}\right)/\left(-\frac{2^2}{3}\right) = -\left(\frac{2^3}{3^2}\right)\left(\frac{3}{2^2}\right) = -\frac{2}{3}.$$

This means $a = -\frac{4}{3}$ and $r = -\frac{2}{3}$. Then $|r| < 1$ so the series converges to

$$\frac{a}{1-r} = \frac{-\frac{4}{3}}{1 - \frac{-2}{3}} = -\frac{4}{5}$$

11. Find the family of orthogonal trajectories to the family of curves given by

$$y = k(\sqrt[3]{x})$$

Solution:
First we compute y'

$$y' = k\frac{1}{3}x^{-2/3} \ .$$

We now solve for k in $y = k(\sqrt[3]{x})$. Doing so we get $k = \frac{y}{\sqrt[3]{x}}$. So

$$y' = \frac{y}{\sqrt[3]{x}}\frac{1}{3}x^{-2/3} = \frac{y}{3x} \ .$$

If $y \neq 0$ we take the negative inverse, we get that the family of orthogonal trajectories satisfies

$$y' = -\frac{3x}{y} \, .$$

We then separate to get

$$y\,y' = -3x$$

Integrating we get

$$\int y\,dy = -\int 3x\,dx \ .$$

So

$$\frac{1}{2}y^2 = -\frac{3}{2}x^2 + C \ .$$

Rearranging we finde

$$y^2 + 3x^2 = \tilde{C}\ .$$

If there is an x such that $y(x) = 0$ then y is identically 0 we have the family

$$y^2 + 3x^2 = \tilde{C} \text{ and } y = 0$$

12. Find the arc length of the curve $y = f(x)$ from the point $(0, 1/3)$ to the point $(1, \frac{e^3+e^{-3}}{6})$ where

$$f(x) = \frac{e^{3x} + e^{-3x}}{6}.$$

Solution:
This curve is a function of x so we can use

$$L = \text{arc length} = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

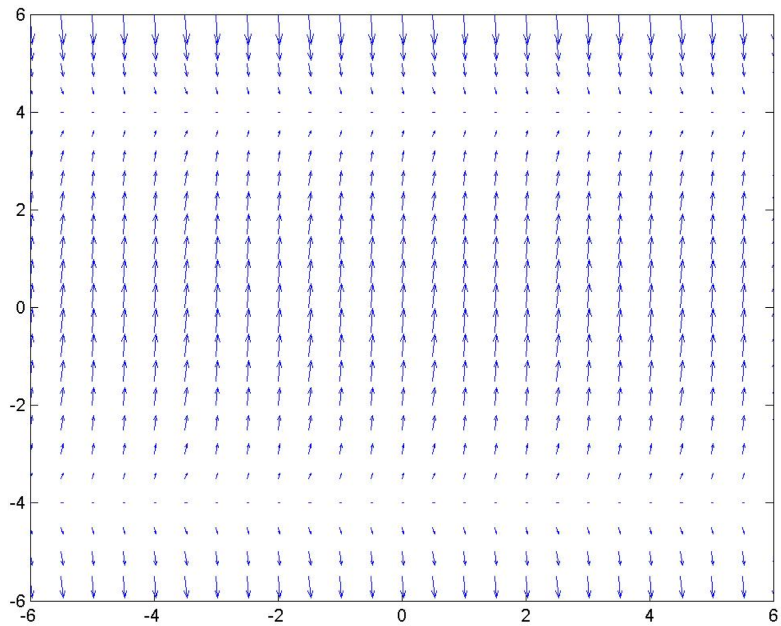
Now $\frac{dy}{dx} = f'(x) = \frac{e^{3x}-e^{-3x}}{2}$, so

$$\begin{aligned}
\text{arc length} &= \int_0^1 \sqrt{1 + \left(\frac{e^{3x} - e^{-3x}}{2}\right)^2}\, dx \\
&= \int_0^1 \sqrt{1 + \left(\frac{e^{6x} - 2 + e^{-6x}}{4}\right)}\, dx \\
&= \int_0^1 \sqrt{\frac{e^{6x} + 2 + e^{-6x}}{4}}\, dx \\
&= \int_0^1 \sqrt{\left(\frac{e^{3x} + e^{-3x}}{2}\right)^2}\, dx \\
&= \int_0^1 \frac{e^{3x} + e^{-3x}}{2}\, dx \\
&= \left.\frac{e^{3x} + e^{-3x}}{6}\right|_0^1 dx \\
&= \frac{e^3 - e^{-3}}{6}
\end{aligned}$$

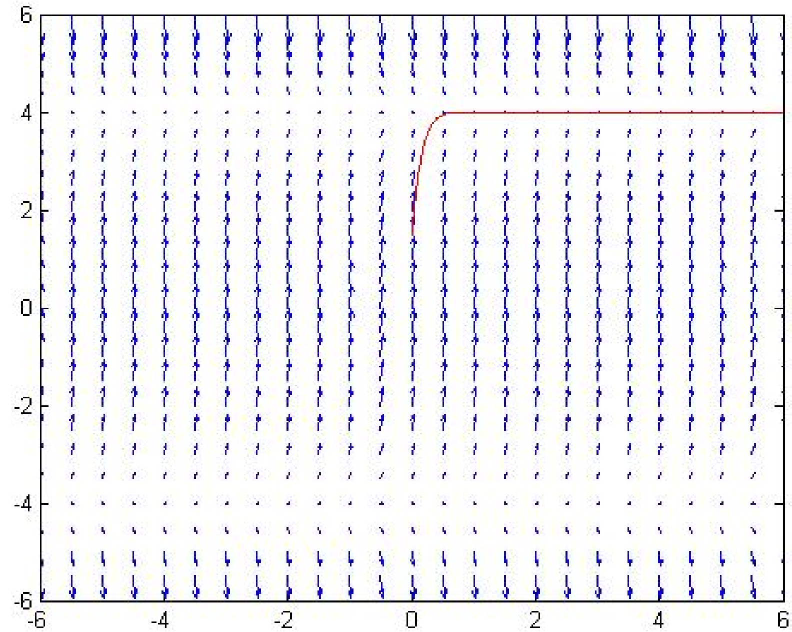
13. (a) Which of the pictures below show the direction field for the differential equation $\frac{dy}{dx} = (4 - y)(4 + y)$.
(b) On the direction field you have selected above , sketch the graph of the solution with initial condition $y(0) = \frac{3}{2}$.

(c) For the solution you have sketched in part (b), use the direction field to determine $\lim_{x\to\infty} y(x)$?

Solution: (a) When $y > 4$, $\frac{dy}{dx} = (4-y)(4+y) < 0$ so the slopes should be negative for all points above $y = 4$. Similarly when $y < -4$, $\frac{dy}{dx} = (4-y)(4+y) < 0$ so all points below $y = -4$ should also be negative. When $-4 < y < 4$, $\frac{dy}{dx} = (4-y)(4+y) > 0$ so all points in between should have positive slope. This is answer (IV).



(b) The point $(0, \frac{3}{2})$ is in the middle portion y should slowly curve up to $y = 4$.



(c) This means that for this initial condition

$$\lim_{x\to\infty} y(x) = 4$$